First Principles Justification of a "Single Wave Model" for Electrostatic Instabilities

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Abstract

The nonlinear evolution of a unstable electrostatic wave is considered for a multi-species Vlasov plasma. From the singularity structure of the associated amplitude expansions, the asymptotic features of the electric field and distribution functions are determined in the limit of weak instability, i.e. $\gamma \to 0^+$ where γ is the linear growth rate. The asymptotic electric field is monochromatic at the wavelength of the linear mode with a nonlinear time dependence. The structure of the distibutions outside the resonant region is given by the linear eigenfunction but in the resonant region the distribution is nonlinear. The details depend on whether the ions are fixed or mobile; in either case the physical picture corresponds to the single wave model originally proposed by O"Neil, Winfrey, and Malmberg for the interaction of a cold weak beam with a plasma of fixed ions.

I. INTRODUCTION

Recently, we have studied the collisionless nonlinear evolution of an unstable mode; first in a single component Vlasov plasma with fixed ions and then more generally in a multi-species Vlasov plasma. [1]- [3] The asymptotic features of the problem in the limit of weak instability, i.e. $\gamma \to 0^+$ where γ is the linear growth rate, were our principal focus. The main tool has been expansions for the amplitude equation and the distribution functions; in particular the asymptotic structure of these expansions. Coefficients of both expansions develop singularities as $\gamma \to 0^+$ and these singularities reveal the asymptotic features of the amplitude equation, distribution functions and electric field.

The amplitude equation describes the evolution on the unstable manifold of the equilibrium and a key conclusion of our previous paper established the scaling behavior of this system. [3] By setting $A(t) = \gamma^\beta r(\gamma t)\exp(-i\theta(t))$, the resulting equations for $r(\tau)$ $(\tau \equiv \gamma t)$ and $\theta(t)$ were free of singular behavior as $\gamma \to 0^+$, provided the exponent β was suitably chosen. The correct choice turned out to depend on the model under consideration: if ion masses were finite, then typically $\beta = 5/2$ unless the ion distributions happened to be flat at the phase velocity of the linear wave. In the limit of fixed ions $(m_i \to \infty)$ or when the ion distributions are flat at the resonant velocity, the exponent drops to $\beta = 1/2$.

In this paper we apply these results for β to control and interpret the singularities that arise in the expansions of the distribution functions. This study illuminates in detail the asymptotic structure of both the distributions and the electric field. In particular, we find that the electric field is essentially monochromatic at the wavelength of the linear mode with a nonlinear time dependence. Outside the resonant region, the distributions are described by the linear eigenfunction and in the resonant region they have a nonlinear structure. The details depend on whether the ions are fixed or mobile, but in either case this physical picture is well known from the "single wave model" proposed by O'Neil, Winfrey and Malmberg for the interaction of cold weak beam with a neutral plasma of mobile electrons and infinitely massive ions.. [4] Their work supplied a model of the self-consistent Vlasov problem that has proven useful to many researchers in the subsequent years. [5]- [12] Our conclusions generalize this useful simplified picture to a general electrostatic instability arising in an unmagnetized multi-species Vlasov plasma. As this paper was being completed, we learned of the interesting recent work by del-Castillo-Negrete who has given an different derivation of the single wave picture using matched asymptotic methods to treat the resonant and non-resonant particles. [13] As in the original work of O'Neil *et al.*, del-Castillo-Negrete allows only mobile electrons and moreover restricts attention to instabilities associated with so-called "inflection point modes". [14,15]

In the remainder of this introduction we review our notation and in section II we summarize the needed conclusions of Ref [3] regarding the singularities of the expansions. The third section applies these conclusions to the distributions and electric field, and section IV contains a final discussion.

A. Notation

Our notation follows Ref [3]; we consider a one-dimensional, multi-species Vlasov plasma defined by

$$\frac{\partial F^{(s)}}{\partial t} + v \frac{\partial F^{(s)}}{\partial x} + \kappa^{(s)}\ E\ \frac{\partial F^{(s)}}{\partial v} = 0 \tag{1}$$

$$\frac{\partial E}{\partial x} = \sum_s \int_{-\infty}^{\infty} dv\, F^{(s)}(x,v,t). \tag{2}$$

Here x, t and v are measured in units of u/ω_e, ω_e^{-1} and u, respectively, where u is a chosen velocity scale and $\omega_e^2 = 4\pi e^2 n_e/m_e$. The plasma length is L with periodic boundary conditions and we adopt the normalization

$$\int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dv\, F^{(s)}(x,v,t) = \left(\frac{z_s\ n_s}{n_e}\right) L \tag{3}$$

where $q_s = e\, z_s$ is the charge of species s and $\kappa^{(s)} \equiv q_s m_e/e m_s$. Note that $\kappa^{(e)} = -1$ for electrons and that the normalization (3) for negative species makes the distribution function negative.

Let $F_0(v)$ and $f(x,v,t)$ denote the multi-component fields for the equilibrium and perturbation respectively and κ the matrix of mass ratios,

$$f \equiv \begin{pmatrix} f^{(s_1)} \\ f^{(s_2)} \\ \vdots \end{pmatrix} \quad F_0 \equiv \begin{pmatrix} F_0^{(s_1)} \\ F_0^{(s_2)} \\ \vdots \end{pmatrix} \quad \kappa \equiv \begin{pmatrix} \kappa^{(s_1)} & 0 & 0 & \cdots \\ 0 & \kappa^{(s_2)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{4}$$

then the system (1) - (2) can be concisely expressed as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + \mathcal{N}(f) \tag{5}$$

where the linear operator is defined by

$$\mathcal{L}\, f = \sum_{l=-\infty}^{\infty} e^{ilx}\, (L_l f_l)(v) \tag{6}$$

$$(L_l f_l)(v) = \begin{cases} 0 & l = 0 \\ -il \left[v f_l(v) + \kappa \cdot \eta_l(v) \sum_s \int_{-\infty}^{\infty} dv'\, f_l^{(s)}(v') \right] & l \neq 0, \end{cases} \tag{7}$$

with $\eta_l(v) \equiv -\partial_v F_0/l^2$, and the nonlinear operator $\mathcal{N}$ is

$$\mathcal{N}(f) = \sum_{m=-\infty}^{\infty} e^{imx} {\sum_{l=-\infty}^{\infty}}' \frac{i}{l} \left(\kappa \cdot \frac{\partial f_{m-l}}{\partial v}\right) \sum_{s'} \int_{-\infty}^{\infty} dv'\, f_l^{(s')}(v'). \tag{8}$$

In the spatial Fourier expansion (6), l denotes an integer multiple of the basic wavenumber $2\pi/L$, and the primed summation in (8) omits the $l = 0$ term. The notation $\kappa \cdot \eta_l(v)$ or $\kappa \cdot \partial_v f_{m-l}$ denotes matrix multiplication. For two multi-component fields of (x,v), e.g. $B = (B^{(s_1)}, B^{(s_2)}, B^{(s_3)}, ..)$ and $D = (D^{(s_1)}, D^{(s_2)}, D^{(s_3)}, ..)$, we define an inner product by

$$(B, D) \equiv \sum_s \int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dv\, B^{(s)}(x,v)^* D^{(s)}(x,v) = \int_{-L/2}^{L/2} dx\ < B, D > \tag{9}$$

where

$$< B, D > \equiv \sum_s \int_{-\infty}^{\infty} dv\, B^{(s)}(x,v)^* D^{(s)}(x,v). \tag{10}$$

The spectral theory for $\mathcal{L}$ is well established and the facts needed for our analysis are easily summarized. The eigenvalues $\lambda = -ilz$ of $\mathcal{L}$ are determined by the roots $\Lambda_l(z) = 0$ of the "spectral function",

$$\Lambda_l(z) \equiv 1 + \int_{-\infty}^{\infty} dv\, \frac{\sum_s \kappa^{(s)} \eta_l^{(s)}(v)}{v - z}. \tag{11}$$

If the contour in (11) is replaced by the Landau contour for $\mathrm{Im}(z) < 0$ then we have the linear dielectric $\epsilon_l(z)$; for $\mathrm{Im}(z) > 0$, $\Lambda_l(z)$ and $\epsilon_l(z)$ are the same function. The eigenvalues can be either real or complex depending on the symmetry and shape of the equilibrium.

Associated with an eigenvalue $\lambda = -ilz$ is the multi-component eigenfunction $\Psi(x,v) = e^{ilx}\,\psi(v)$ where

$$\psi(v) = -\frac{\kappa \cdot \eta_l}{v - z}. \tag{12}$$

There is also an associated adjoint eigenfunction $\tilde{\Psi}(x,v) = e^{ilx}\tilde{\psi}(v)/L$ satisfying $(\tilde{\Psi}, \Psi) = 1$ with

$$\tilde{\psi}(v) = -\frac{1}{\Lambda_l'(z)^*(v - z^*)}. \tag{13}$$

Note that all components of $\tilde{\psi}(v)$ are the same. The normalization in (13) assumes that the root of $\Lambda_l(z)$ is simple and is chosen so that $< \tilde{\psi}, \psi >= 1$. The adjoint determines the projection of $f(x,v,t)$ onto the eigenvector, and this projection defines the time-dependent amplitude of Ψ, i.e. $A(t) \equiv (\tilde{\Psi}, f)$.

II. PREVIOUS RESULTS

The equilibrium $F_0(v)$ is assumed to support a "single" unstable mode in the sense that E^u, the unstable subspace for $\mathcal{L}$, is two-dimensional. With translation symmetry and periodic boundary conditions, this is the simplest instability problem that can be posed. Henceforth, let k denote the wavenumber of this unstable mode that is associated with the root $\Lambda_k(z_0) = 0$ which we assume to be simple, i.e. $\Lambda_k'(z_0) \neq 0$. The corresponding eigenvector is

$$\Psi(x,v) = e^{ikx}\,\psi(v) = e^{ikx}\left(-\frac{\kappa \cdot \eta_k}{v - z_0}\right). \tag{14}$$

The root $z_0 = v_p + i\gamma/k$ determines the phase velocity $v_p = \omega/k$ and the growth rate γ of the linear mode as the real and imaginary parts of the eigenvalue $\lambda = -ikz_0 = \gamma - i\omega$.

Solutions on the unstable manifold have the form

$$f^u(x,v,t) = \left[A(t)\psi(v)e^{ikx} + A^*(t)\psi^*(v)e^{-ikx}\right] + H(x,v,A(t),A^*(t)) \tag{15}$$

where $A(t) \equiv (\tilde{\Psi}, f^u)$ evolves according to the amplitude equation

$$\dot{A} = \lambda\, A + (\tilde{\Psi}, \mathcal{N}(f^u)) \tag{16}$$

and self-consistency requires H to satisfy

$$\frac{\partial H}{\partial A}\dot{A} + \frac{\partial H}{\partial A^*}\dot{A}^* = \mathcal{L}H + \mathcal{N}(f^u) - \left[(\tilde{\Psi}, \mathcal{N}(f^u))\,\Psi + cc\right] \tag{17}$$

subject to the geometric constraints

$$0 = H(x, v, 0, 0) = \frac{\partial H}{\partial A}(x, v, 0, 0) = \frac{\partial H}{\partial A^*}(x, v, 0, 0). \tag{18}$$

The translation symmetry of the model (5) provides important constraints on both the amplitude equation and the form of H. [3] For the amplitude equation (16), the right hand side must have the form

$$\lambda A + (\tilde{\Psi}, \mathcal{N}(f^u)) = Ap(\sigma) \tag{19}$$

where $\sigma \equiv |A|^2$ and $p(\sigma)$ is an unknown function to be determined from the model. Similarly, translational symmetry requires the spatial Fourier components of H to have a special form

$$\begin{aligned} H_0(v, A, A^*) &= \sigma\, h_0(v, \sigma) \\ H_k(v, A, A^*) &= A\sigma\, h_1(v, \sigma) \\ H_{mk}(v, A, A^*) &= A^m\, h_m(v, \sigma) \quad \text{for } m \geq 2 \end{aligned} \tag{20}$$

where $H_{-l} = {H_l}^*$. These results focus our analysis on a set of functions, $\{p(\sigma), h_m(v, \sigma)\}$, which must be determined from the Vlasov equation.

A. Expansions and singularities

We study $p(\sigma)$ and $\{h_m(v, \sigma)\}$ via the expansions

$$p(\sigma) = \sum_{j=1}^{\infty} p_j \sigma^j \qquad\qquad h_m(v, \sigma) = \sum_{j=1}^{\infty} h_{m,j}(v)\sigma^j. \tag{21}$$

The coefficients p_j and $h_{m,j}$ are determined by inserting the expansions into (17) and (19) and solving at each order of σ. The resulting recursion relations are given in Ref [3] and are not required for the present discussion.

The key point is that for both the amplitude equation and the distribution function the expansion coefficients develop singularities in the limit $\gamma \to 0^+$. This can be seen explicitly by reviewing the calculation of the cubic coefficient p_1. From Ref [3], p_1 depends on $h_{0,0}$ and $h_{2,0}$,

$$p_1 = -\frac{i}{k}\left[< \partial_v\tilde{\psi}, \kappa \cdot (h_{0,0} - h_{2,0}) > + \frac{\Gamma_{2,0}}{2} < \partial_v\tilde{\psi}, \kappa \cdot \psi^* > \right], \tag{22}$$

where $\Gamma_{2,0} = \int dv\, h_{2,0}$. The recursion relations determine $h_{0,0}$ and $h_{2,0}$,

$$h_{0,0}(v) = -\frac{1}{k^2}\frac{\partial}{\partial v}\left[\frac{\kappa^2 \cdot \eta_k}{(v-z_0)(v-z_0^*)}\right] \tag{23}$$

$$h_{2,0}(v) = \frac{1}{2k^2}\left(\frac{\kappa \cdot \partial_v \psi}{v-z_0}\right) + \frac{1}{6k^2}\left(\frac{\kappa \cdot \eta_k}{v-z_0}\right)\left(\frac{\kappa \cdot \eta_k}{v-z_0}\right), \tag{24}$$

and one notes that for $\gamma > 0$ these are smooth functions but there are complex poles at z_0 and z_0^* that approach the real axis at $v = v_p$ in the limit $\gamma \to 0^+$. For $h_{2,0}$ all poles lie above the real axis, but $h_{0,0}$ contains poles above and below the axis and this forces the integral $< \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} >$ in p_1 to diverge as $\gamma \to 0^+$ because of a pinching singularity. For similar reasons, the integral $< \partial_v \tilde{\psi}, \kappa \cdot \psi^* >$ also diverges but the remaining integrals in p_1 are nonsingular.

A detailed evaluation of this asymptotic structure in p_1 yields the form

$$p_1 = \frac{1}{\gamma^4}\left[c_1(\gamma) - \gamma\, d_1(\gamma) + \mathcal{O}(\gamma^2)\right] \tag{25}$$

where c_1 and d_1 are nonsingular functions of γ defined by

$$c_1(\gamma) = -\frac{k}{4\Lambda_k'(z_0)}{\sum_s}' \kappa^{(s)}(1-\kappa^{(s)2})\ \mathrm{Im}\left(\int_{-\infty}^{\infty} dv \frac{\eta_k^{(s)}}{v-z_0}\right) \tag{26}$$

$$d_1(\gamma) = \frac{1}{4} - \frac{1}{4\Lambda_k'(z_0)}{\sum_s}' \kappa^{(s)}(1-\kappa^{(s)2}) \int_{-\infty}^{\infty} dv \frac{\eta_k^{(s)}}{(v-z_0)^2} \tag{27}$$

where the primed species sum omits the electrons. At $\gamma = 0$, c_1 has the limit

$$c_1(0) = -\frac{\pi k}{4\Lambda_k'(z_0)}{\sum_s}' \kappa^{(s)}(1-\kappa^{(s)2})\eta_k^{(s)}(v_p) \tag{28}$$

which is typically non-zero yielding a γ^{-4} singularity for p_1. There are at least two special cases of interest for which $c_1(0) = 0$; namely, infinitely massive fixed ions ($\kappa^{(s)} = 0$ for all $s \neq e$) and flat ion distributions at the resonant velocity ($\eta_k^{(s)}(v_p) = 0$ for all $s \neq e$). In such cases, the divergence of p_1 drops to γ^{-3}.

Analagous singularities appear also in the higher order coefficients and grow more severe although their character remains the same. The higher coefficients $h_{m,j}$ exhibit more and more poles which approach the linear phase velocity as $\gamma \to 0^+$ and these poles generate stronger pinching singularities in the higher coefficients p_j. An important property of the poles in $h_{m,j}$ is that they always have the general form $(v-\alpha)^{-n}$ or $(v-\alpha^*)^{-n}$ with

$$\alpha = z_0 + i\gamma\zeta/k = v_p + i\gamma(\zeta+1)/k \tag{29}$$

where $\zeta > 0$ is a purely numerical factor, i.e. the poles always lie along the vertical line $\mathrm{Re}(v) = v_p$.

The explicit calculation of higher order coefficients from recursion relations rapidly becomes prohibitively laborious; however, useful bounds on the singularity of the higher order coefficients are obtained Ref [3] using an induction argument. More precisely, we find for the amplitude equation

$$\lim_{\gamma\to 0^+} \gamma^{\nu}|p_j| < \infty \tag{30}$$

for $j \geq 1$ where $\nu = 5j - 1$ in the generic case with $c_1(0) \neq 0$, and $\nu = 4j - 1$ in the two special cases mentioned above, fixed ions or flat ion distributions, with $c_1(0) = 0$. For the coefficients of the distribution function, the induction argument proves, for $m \geq 0$, $j \geq 0$ and $m' \geq 0$,

$$\lim_{\gamma\to 0^+} \gamma^{\mu_{m,j}} \left| \int_{-\infty}^{\infty} dv \sum_s \left(\kappa^{(s)}\right)^{m'} h^{(s)}_{m,j}(v) \right| < \infty \tag{31}$$

where $\mu_{m,j} = J_{m,j} + 1$ with $J_{m,j} \equiv (2m + 5j - 3) + 4\delta_{m,0} + 5\delta_{m,1}$ in the generic case defined by $c_1(0) \neq 0$. For the special cases of fixed ions or flat ion distributions, (31) holds with exponent $\mu_{m,j} = J_{m,j} - j - \delta_{m,1}$.

The first bound determines the scaling exponent for A. When (16), (19), and (21) are combined we obtain an amplitude equation,

$$\dot{A} = \lambda A + \sum_{j=1}^{\infty} p_j |A|^{2j} A \tag{32}$$

where each nonlinear term has a singular coefficient and the equation is ill-defined as $\gamma \to 0^+$. The cure is to rescale the amplitude

$$A(t) \equiv \gamma^{\beta}\, r(\gamma t) e^{-i\theta(t)} \tag{33}$$

with $\beta = 5/2$ for the typical case ($c_1(0) \neq 0$) and in the special cases with $c_1(0) = 0$ we require $\beta = 2$. Once this is done, the equations for $r(\tau)$ and $\theta(t)$ are nonsingular in the regime of weak growth rates; additional details may be found in Ref [3].

III. DISTRIBUTION FUNCTIONS AND ELECTRIC FIELD

The scaling (33) of the amplitude has immediate implications for the asymptotic structure of the distributions. From (15) the Fourier coefficients of $f = F - F_0$ may be written in terms of $r(\tau)$ and $\theta(t)$ in (33) and h_m

$$\begin{aligned} f_0(v,t) &= \gamma^{2\beta}\, r(\tau)^2\, h_0(v, \gamma^{2\beta} r^2) \\ f_k(v,t) &= \gamma^{\beta}\, r(\tau) e^{-i\theta(t)}\, \left[\psi(v) + \gamma^{2\beta}\, r(\tau)^2\, h_1(v, \gamma^{2\beta} r^2)\right] \\ f_{mk}(v,t) &= \gamma^{2m\beta}\, r(\tau)^m\, e^{-im\theta(t)}\, h_m(v, \gamma^{2\beta} r^2) \qquad\qquad m \geq 2. \end{aligned} \tag{34}$$

As $\gamma \to 0^+$, $r(\tau)$ is an $\mathcal{O}(1)$ quantity, thus the asymptotic features of each Fourier component are determined by the explicitly shown factors of γ and the asymptotic form of the functions $\psi(v)$ and $h_m(v, \gamma^{2\beta} r^2)$. The dependence on h_m necessitates a separate consideration of the asymptotic behavior for non-resonant and resonant velocities. In the former regime we assume the distance from the linear phase velocity satisfies $v - v_p = \mathcal{O}(1)$, and the resonant regime corresponds to velocities within a neighborhood of v_p that scales with the growth rate, i.e. $v - v_p = \mathcal{O}(\gamma)$. For resonant velocities, the singularities in $\psi(v)$ and $h_m(v, \gamma^{2\beta} r^2)$ come into play and alter the asymptotic features of the distribution function.

A. Non-resonant velocities

For $v - v_p = \mathcal{O}(1)$, the functions $\psi(v)$ and $h_m(v,\sigma)$ are bounded $\mathcal{O}(1)$ quantities (we use $\sigma = \gamma^{2\beta} r^2$ to emphasize this), and the Fourier components (34) combine to yield

$$\begin{aligned}(F(x,v,t) - F_0(v))/\gamma^\beta = & \left[r(\tau)e^{-i\theta(t)}\Psi(x,v) + \mathrm{cc}\right] + \gamma^\beta\, r(\tau)^2\, h_0(v,\sigma) \\ & + \gamma^{2\beta} r(\tau)^3 \left[e^{-i\theta(t)} h_1(v,\sigma) e^{ikx} + \mathrm{cc}\right] \\ & + \sum_{m=2}^{\infty} \left[\gamma^{(2m-1)\beta} e^{-im\theta(t)}\, h_m(v,\sigma) e^{imkx} + \mathrm{cc}\right] \end{aligned} \tag{35}$$

$$\approx \left[r(\tau)e^{-i\theta(t)}\Psi(x,v) + \mathrm{cc}\right] + \mathcal{O}(\gamma^\beta). \tag{36}$$

In words, the non-resonant correction to F_0 scales overall as γ^β; the leading piece of this correction simply has the form of the linear wave $\Psi(x,v)$ with nonlinear time dependence determined by the mode amplitude $r(\tau)\exp(-i\theta(t))$.

B. Resonant velocities

For $v - v_p = \mathcal{O}(\gamma)$, the functions $\psi(v)$ and $h_m(v,\sigma)$ typically develop singularities as $\gamma \to 0^+$ and these divergences compete with the explicit factors of γ in (34) to determine the asymptotic form of the distribution. The analysis is simplified by the fact that all relevant singularities are poles of the form described in (29), and these may be rewritten as a singular factor multiplying a nonsingular function of the rescaled velocity variable $u \equiv (v - v_p)/\gamma$, e.g.

$$\frac{1}{(v-\alpha)^n} = \frac{1}{\gamma^n} \frac{1}{((v-v_p)/\gamma - i(\zeta+1)/k)^n} = \frac{1}{\gamma^n} \frac{1}{((u - i(\zeta+1)/k)^n}. \tag{37}$$

Once this is done, the functions $\psi(v)$ and $h_m(v,\sigma)$, expressed in terms of u, may be substituted into (34); the variable u provides a uniform velocity coordinate for the resonant region.

The puzzle is to deduce the correct overall factor of $1/\gamma^n$ for each function. For $h_m(v,\sigma)$ we have the integral bound (31) which may be rewritten in terms of u

$$\lim_{\gamma\to 0^+} \left| \int_{-\infty}^{\infty} du \sum_s \left(\kappa^{(s)}\right)^{m'} \gamma^{1+\mu_{m,j}} h_{m,j}(v_p + \gamma u) \right| < \infty. \tag{38}$$

Since all singularities are poles we know the integrand does not have an integrable singularity, hence we conclude that $\gamma^{1+\mu_{m,j}} h_{m,j}(v_p + \gamma u)$ defines a nonsingular function of u:

$$h_{m,j}(v_p + \gamma u) \equiv \gamma^{-(1+\mu_{m,j})} \hat{h}_{m,j}(u,\gamma). \tag{39}$$

The nonsingular character of $\hat{h}_{m,j}(u,\gamma)$ can be checked directly for the specific examples in (23) and (24), and also verified, in general, from the recursion relations. From the expansion of $h_m(v,\sigma)$, we thus find

$$h_m(v_p+\gamma u,\sigma) = \sum_{j=0}^{\infty} \gamma^{2j\beta-(1+\mu_{m,j})} \hat{h}_{m,j}(u,\gamma) r^{2j}. \tag{40}$$

In the generic case with $\beta = 5/2$ and $\mu_{m,j} = J_{m,j} + 1$ this gives

$$h_m(v_p+\gamma u,\sigma) = \frac{1}{\gamma^{\delta_m}} \sum_{j=0}^{\infty} \hat{h}_{m,j}(u,\gamma) r^{2j} \tag{41}$$

where

$$\delta_m = \begin{cases} 3 & m=0 \\ 6 & m=1 \\ 2m-1 & m\geq 2 \end{cases} \qquad (c_1(0)\neq 0). \tag{42}$$

In the special cases with fixed ions or flat distributions, then $\beta = 2$ and $\mu_{m,j} = J_{m,j} - j - \delta_{m,1}$, and (41) holds with exponent

$$\delta_m = \begin{cases} 2 & m=0 \\ 4 & m=1 \\ 2m-2 & m\geq 2 \end{cases} \qquad (c_1(0) = 0). \tag{43}$$

In all cases, we define the nonsingular function $\hat{h}_m(u,r^2,\gamma) \equiv \sum_{j=0}^{\infty} \hat{h}_{m,j}(u,\gamma) r^{2j}$ and rewrite (41)

$$h_m(v_p+\gamma u,\sigma) = \frac{\hat{h}_m(u,r^2,\gamma)}{\gamma^{\delta_m}}. \tag{44}$$

It is simpler to obtain the corresponding factorization of the eigenfunction; from the definition (14) we have

$$\psi(v_p+\gamma u) = \frac{1}{\gamma}\left(-\frac{\kappa\cdot\eta_k(v_p+\gamma u)}{u-i/k}\right), \tag{45}$$

and the only subtlety concerns $\eta_k(v_p+\gamma u)$ which is $\mathcal{O}(1)$ in the generic case ($c_1(0)\neq 0$) and $\mathcal{O}(\gamma)$ in the two special cases with $c_1(0) = 0$. Thus we define the nonsingular function $\hat{\psi}(u,\gamma)$ by

$$\psi(v_p+\gamma u) = \frac{\hat{\psi}(u,\gamma)}{\gamma} \qquad (c_1(0)\neq 0), \tag{46}$$

in the generic case, but in the special cases the eigenfunction is itself nonsingular and we have

$$\psi(v_p+\gamma u) = \hat{\psi}(u,\gamma) \qquad (c_1(0) = 0). \tag{47}$$

We are now able to describe the asymptotic structure of the distributions.

1. *Generic instability:* $c_1(0) \neq 0$

For the generic case, inserting (44) and (46) into (34) yields

$$[F(x, v_p + \gamma u, t) - F_0(v_p + \gamma u)]/\gamma^{3/2} = \left\{ r(\tau)\, e^{-i\theta(t)}\, e^{ikx} \left[\hat{\psi}(u, \gamma) + r(\tau)^2\, \hat{h}_1(u, r^2, \gamma) \right] + \mathrm{cc} \right\}$$
$$+ \sqrt{\gamma} \left\{ r(\tau)^2\, \hat{h}_0(u, r^2, \gamma) + \sum_{m=2}^{\infty} \left[\gamma^{(m-2)/2}\, e^{imkx}\, r(\tau)^m\, e^{-im\theta(t)}\, \hat{h}_m(u, r^2, \gamma) + \mathrm{cc} \right] \right\}; \quad (48)$$

neglecting the subdominant terms this gives

$$\left[F(x, v_p + \gamma u, t) - F_0(v_p + \gamma u) \right] / \gamma^{3/2} = \left\{ r(\tau)\, e^{-i\theta(t)}\, e^{ikx} \left[\hat{\psi}(u, \gamma) + r(\tau)^2\, \hat{h}_1(u, r^2, \gamma) \right] + \mathrm{cc} \right\}$$
$$+ \mathcal{O}(\sqrt{\gamma}). \quad (49)$$

The generic resonant correction to F_0, expressed in the velocity coordinate u, scales overall as $\gamma^{3/2}$; the leading term in this correction has the wavelength of the linear wave but the velocity dependence, $\hat{\psi}(u, \gamma) + r(\tau)^2\, \hat{h}_1(u, r^2, \gamma)$, is not simply given by the linear eigenfunction. The time dependence is determined by the mode amplitude $r(\tau) \exp(-i\theta(t))$ but the dependence on r is rather complicated.

2. *Special cases:* $c_1(0) = 0$

For the special cases, defined by fixed ions or flat ion distributions, we apply (47) and (43)-(44) to (34) and obtain

$$[F(x, v_p + \gamma u, t) - F_0(v_p + \gamma u)]/\gamma^2 = \left\{ r(\tau)\, e^{-i\theta(t)}\, e^{ikx} \left[\hat{\psi}(u, \gamma) + r(\tau)^2\, \hat{h}_1(u, r^2, \gamma) \right] + \mathrm{cc} \right\}$$
$$+ r(\tau)^2\, \hat{h}_0(u, r^2, \gamma) + \sum_{m=2}^{\infty} \left[e^{imkx}\, r(\tau)^m\, e^{-im\theta(t)}\, \hat{h}_m(u, r^2, \gamma) + \mathrm{cc} \right]. \quad (50)$$

This is a qualitatively different structure in contrast to (48); now the resonant correction is $\mathcal{O}(\gamma^2)$ and *all* wavelengths are present at leading order. Thus the spatial dependence is very rich and bears no special relation to the linear instability; a similar observation holds for the dependence on velocity.

C. Electric field

The Fourier components of E are given by (34) and Poisson's equation

$$ikE_k(t) = \gamma^{\beta}\, r(\tau) e^{-i\theta(t)} \left[1 + \gamma^{2\beta}\, r(\tau)^2 \int_{-\infty}^{\infty} dv \sum_s h_1^{(s)}(v, \sigma) \right]$$
$$imkE_{mk}(t) = \gamma^{2m\beta}\, r(\tau)^m\, e^{-im\theta(t)} \int_{-\infty}^{\infty} dv \sum_s h_m^{(s)}(v, \sigma) \qquad m \geq 2. \quad (51)$$

Bounds on the asymptotic form of the integrals can be inferred from the expansion $h_m = \sum_j h_{m,j} \sigma^j$ and the bound (31) on the integrals of $h_{m,j}$ (for $m' = 0$). The details depend on whether we consider the generic instability or the special cases.

1. *Generic instability:* $c_1(0) \neq 0$

From (31) we find

$$\lim_{\gamma\to 0^+} \gamma^{\alpha_m} \left| \int_{-\infty}^{\infty} dv \sum_s h_m^{(s)}(v) \right| < \infty \tag{52}$$

with

$$\alpha_m = \begin{cases} 2 & m = 0 \\ 5 & m = 1 \\ 2m-2 & m \geq 2 \end{cases} \qquad (c_1(0) \neq 0). \tag{53}$$

Hence, with $\beta = 5/2$, the generic components are

$$\begin{aligned} ikE_k(t) &= \gamma^{5/2}\, r(\tau) e^{-i\theta(t)} \left[1 + r(\tau)^2\, \gamma^5 \int_{-\infty}^{\infty} dv \sum_s h_1^{(s)}(v,\sigma) \right] \\ imkE_{mk}(t) &= \gamma^{3m+2}\, r(\tau)^m\, e^{-im\theta(t)}\, \gamma^{2m-2} \int_{-\infty}^{\infty} dv \sum_s h_m^{(s)}(v,\sigma) \qquad m \geq 2. \end{aligned} \tag{54}$$

The asymptotic electric field is

$$\frac{E(x,t)}{\gamma^{5/2}} = \frac{1}{k} \left\{ -ir(\tau) e^{-i\theta(t)} \left[1 + r(\tau)^2\, \gamma^5 \int_{-\infty}^{\infty} dv \sum_s h_1^{(s)}(v,\sigma) \right] e^{ikx} + \mathrm{cc} \right\} + \mathcal{O}(\gamma^{11/2}); \tag{55}$$

clearly E is dominated by the wavenumber of the unstable mode with an overall scaling of $\gamma^{5/2}$. The term $\gamma^5 \int dv \sum_s h_1^{(s)}$ is treated as an $\mathcal{O}(1)$ contribution in light of the estimate (52) above.

2. *Special cases:* $c_1(0) = 0$

For instabilities with fixed ions or flat ion distributions, we have $\beta = 2$ and $\mu_{m,j} = J_{m,j} - j - \delta_{m,1}$ in (31); applying this bound to the series $h_m = \sum_j h_{m,j} \sigma^j$ yields

$$\lim_{\gamma\to 0^+} \gamma^{\alpha_m} \left| \int_{-\infty}^{\infty} dv \sum_s h_m^{(s)}(v) \right| < \infty \tag{56}$$

with

$$\alpha_m = \begin{cases} 1 & m = 0 \\ 3 & m = 1 \\ 2m-3 & m \geq 2 \end{cases} \qquad (c_1(0) = 0). \tag{57}$$

Now the general expressions for the components reduce to

$$\begin{aligned} ikE_k(t) &= \gamma^{2}\, r(\tau) e^{-i\theta(t)} \left[1 + \gamma\, r(\tau)^2\, \gamma^3 \int_{-\infty}^{\infty} dv \sum_s h_1^{(s)}(v,\sigma) \right] \\ imkE_{mk}(t) &= \gamma^{2m+3}\, r(\tau)^m\, e^{-im\theta(t)}\, \gamma^{2m-3} \int_{-\infty}^{\infty} dv \sum_s h_m^{(s)}(v,\sigma) \qquad m \geq 2, \end{aligned} \tag{58}$$

and the asymptotic electric field has the form

$$\frac{E(x,t)}{\gamma^2} = \frac{1}{k}\left\{-ir(\tau)e^{-i\theta(t)}\ \left[1+\mathcal{O}(\gamma)\right]\ e^{ikx} + \mathrm{cc}\right\} + \mathcal{O}(\gamma^5). \tag{59}$$

The overall scaling is now the well known γ^2 or "trapping scaling" and the leading term has a much simpler structure. Again we find the wavenumber k of the linear instability; however now the time dependence is simply given by the mode amplitude $r(\tau)\exp(-i\theta(t))$.

IV. DISCUSSION

The single wave model, derived originally by O'Neil, Winfrey and Malmberg, described the interaction of a cold electron beam interacting with a plasma of mobile electrons and fixed ions. In their problem, the infinite extent of the plasma allowed for continuous wavenumbers and the dispersion relation for a cold beam was required to select a single wavenumber corresponding to the maximum growth rate. This wavenumber characterizes the electric field whose nonlinear time development results from the coupling to resonant particles. The nonresonant plasma simply provides a linear dielectric which supports the wave.

By contrast, we pose a more general problem, allowing for multiple mobile species and not restricting the type of electrostatic instability, but for a finite plasma with periodic boundary conditions. Within this setting, we consider equilibria supporting a single unstable mode and derive the resulting equations for the electric field and distributions in the limit of weak instability. In this asymptotic limit, the physical picture of the original single wave model emerges quite generally. The monochromatic electric field is coupled to the resonant particles and evolves nonlinearly while the nonresonant particles show only a linear response to the electric field.

The amplitude expansions, whose singularity structure form the basis of our analysis, do not provide a practical tool for solving the single wave model. For this purpose, it is more convenient to assume the simplifications of the single wave picture and derive model equations directly from the original Vlasov theory. This development will be presented in a forthcoming paper. [16]

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